

KNOTTED SYMMETRIC GRAPHS

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ABSTRACT. For a knotted graph in S^3 we define the *vertex constant group*, a quotient of the fundamental group of the complement. For planar graphs the group is cyclic. For graphs with periodic symmetry the group is related to the fundamental group of the branched cover of S^3 branched over knots contained in the quotient of the graph under the symmetry. These tools are used to demonstrate that a large family of knotted graphs are not planar.

This note presents a new tool for distinguishing knotted graphs in S^3 . To each such graph Γ we associate a group, called the *vertex constant group*; for connected planar graphs the vertex constant group is finite cyclic. Our first application is to offer a new proof that all elements of a particular family of knotted graphs, denoted ϑ_n , are nontrivial. This family of graphs was introduced by Suzuki [Su], expanding on an example of Kinoshita [Ki]; it is of particular interest as a test case for new methods since each element is Brunnian—that is, every subgraph is planar. Suzuki proved that the ϑ_n are nontrivial using an algebraic argument in which he showed that the fundamental group of the complement is not free; Scharlemann [S] has described a more geometric approach, reducing the problem to a calculation in the braid group.

Our second application is to demonstrate that the ϑ_n represents a special case of a large class of knotted graphs, all of which will be proved nontrivial without any explicit calculation. Neither of the approaches of [S, Su] nor apparently any other known techniques apply to the entire class. (See [K1, K2, T, Y] for recent progress in the study of knotted graphs.) Figure 1 illustrates the graph ϑ_5 . One vertex has been placed at the point at infinity to highlight the periodic symmetry of the graph. Similarly, ϑ_n has two vertices joined by n edges. Note that there is a period n transformation, T , of S^3 leaving ϑ_n invariant. This action determines a branched cover of S^3 , the image of ϑ_n , is an arc, K . Figure 2 illustrates K along with the branch set, described as the union of two arcs, B_1 and B_2 . The union of B_2 and K is a trefoil knot. We will see that if the edges of ϑ_n are replaced with any other periodic family of edges, the graph continues to be nontrivial, as long as the union of B_2 and K forms a nontrivial knot in S^3 .

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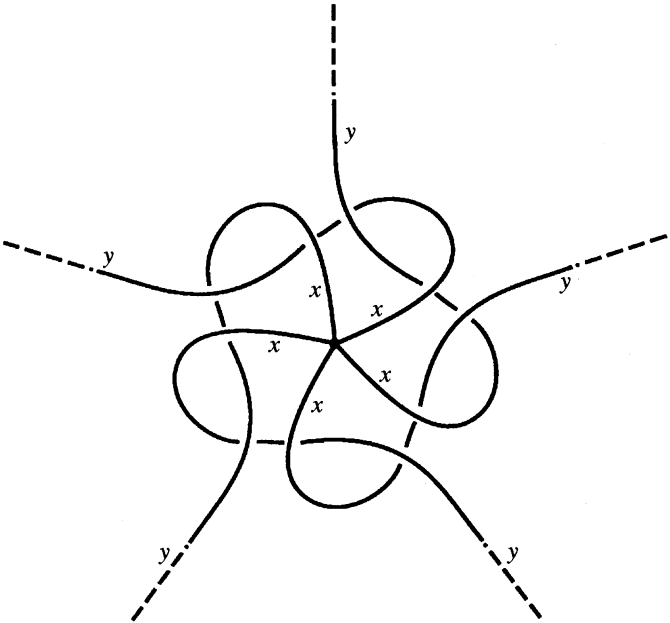


FIGURE 1

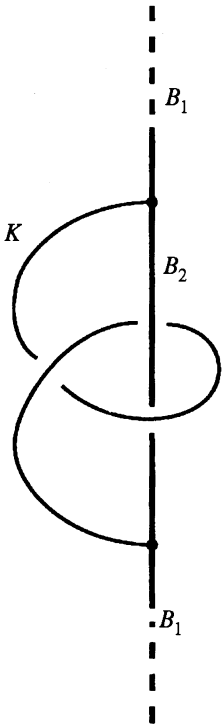


FIGURE 2

1. THE VERTEX CONSTANT GROUP

Given a regular projection of a knotted graph Γ there is an associated Wirtinger presentation of the fundamental group of the complement, π , defined as for knots. The vertex constant group, π^* , is the quotient of π defined by setting all meridians at each vertex equal. For this to be well defined an orientation of each meridian near each vertex must be chosen. This is done by (locally) orienting each edge of the graph near the vertex to point toward that vertex. As one example, Figure 1 includes Wirtinger generators of the graph ϑ_5 . The generators at each vertex have been identified and labelled x and y . A quick calculation shows the vertex constant group has presentation

$$G_n = \langle x, y | xy^{-1}x^{-1} = yxy^{-1}, x^n = 1 \rangle.$$

(For details on the notation of the figure, and of the method of calculation, see [C].)

Proving that the isomorphism class of π^* depends only on the isotopy class of Γ is a standard exercise in knot theory. Note however that the vertex constant group is not an invariant of the complement of the graph, or even of the topological pair (S^3, Γ) ; adding a vertex to an edge adds the relation that the square of the meridian to that edge is trivial.

For a connected planar graph the vertex constant group is easily seen to be cyclic; more specifically, if ϑ_n were planar, then G_n would be isomorphic to Z_n . Here is a quick way to prove this is not the case. Observe that G_n is isomorphic to the group of the trefoil knot with the added relation that the n th power of the meridian is trivial. (In the usual presentation of the trefoil group y is replaced with y^{-1} in the above presentation.) There is a natural homomorphism onto Z_n ; the kernel is the fundamental group of the n -fold cover of the trefoil complement with the added relation that the lift of the meridian to that cover is trivial. That is, the kernel of the map onto Z_n is the fundamental group of the n -fold branched cover of the trefoil. This group is known to be nontrivial. (For the trefoil this can be shown explicitly; it also follows from the Smith Conjecture [B].)

2. PERIODIC GRAPHS

We now want to see that replacing the edges of ϑ_n with a different periodic set of edges will generally leave the graph nonplanar. We begin by offering a slightly different proof that ϑ_n is nontrivial, one that exploits the periodicity of the graph, and then note that this alternative proof immediately extends to the more general setting.

There is a period n transformation, T , on the complement of ϑ_n with fixed set a pair of arcs, B_1^* and B_2^* . The quotient of this action is the complement of the arc K in S^3 , illustrated in Figure 2. The images of the branch curves in the quotient are also shown in Figure 2, labelled B_1 and B_2 . Note that $K \cup B_2$ is the trefoil knot. As will soon be seen, the appearance of the trefoil knot and the role of the trefoil in the previous argument are not coincidental.

Pick a basepoint for $S^3 - \vartheta_n$ on the branch set, B_1^* . Then T defines an action on π denoted T_* . Define a quotient of π , $\pi^\#$, by adding the relations $x = T_*(x)$ for all x in π . Clearly $\pi^\#$ is a quotient of π^* , so showing that $\pi^\#$ is noncyclic will imply that π^* is noncyclic. A covering space argument

presented in the next section shows that $\pi^\#$ is isomorphic to the group of the complement of the knot $K \cup B_2$ with the added relation that the n th power of the meridian to B_2 is trivial. But, as in the argument in the previous section, this group has an index n subgroup isomorphic to the fundamental group of the n -fold branch cover of S^3 branched over $K \cup B_2$. Hence, by the Smith Conjecture, ϑ_n is nontrivial.

This proof just given reveals that the essential property of the ϑ_n is that $K \cup B_2$ is knotted. Hence, if the edges of ϑ_n are replaced with any other periodic family, the graph remains nontrivial, as long as $K \cup B_2$ is nontrivial.

3. COVERING SPACES

We conclude with the covering space argument used in the previous section to analyze $\pi^\#$.

Suppose that $p : (M, B_1^*, \dots, B_k^*) \rightarrow (N, B_1, \dots, B_k)$ is an n -fold cyclic branched covering of 3-manifolds, branched over arcs $B_1, \dots, B_n$, each of branch index n . Let T denote the generator of deck transformations, and let $\pi^\#$ denote the quotient of $\pi_1(M, b_1^*)$ under the relations $x = T_\#(x)$ for all x , where b_1^* is a basepoint on B_1^* .

Theorem. *The group $\pi^\#$ is isomorphic to the quotient of $\pi_1(N - (B_2 \cup \dots \cup B_n), b_1)$ under the relations $m_i^n = 1$, $i > 1$, where the m_i are the meridians to the B_i .*

Proof. First consider the case that $k = 1$. The projection map from $\pi_1(M, b_1^*)$ to $\pi_1(N, b_1)$ is easily seen to determine a well-defined map of the quotient $\pi^\#$ to $\pi_1(N, b_1)$. Next observe that a lifting argument implies that this induced map is surjective. To prove injectivity, represent an element in the kernel by a path γ^* . By transversality it can be assumed that γ^* is disjoint from B_1^* except at its endpoints. Let γ be the image of γ^* in N . Again using transversality it can be assumed that γ bounds a disk in N that meets B_1 transversely, and after an isotopy, all points of intersection of the disk with B_1 are at b_1 . Hence, γ is homotopic, via a homotopy missing B_1 to a product of paths, based at b_1 , each of the form $\omega m_1 \omega^{-1}$ where ω is a closed path in N meeting B_1 only at its endpoints. This homotopy lifts to the branched cover. The lift of each individual path is of the form $x T_\#(x^{-1})$. (The element x is represented by a lift of ω .) Hence, γ^* represents the trivial element in the quotient group $\pi^\#$.

In the case that $k > 1$, consider the manifolds constructed by removing all the components of the branch set of M , other than B_1 , and the corresponding branch curves from the quotient. The argument of the previous paragraph applies for these subspaces. The full spaces are reconstructed by adding in the branch curves. Since the meridians of the branch curves in the cover are identified, via the isomorphism, with the n th powers of meridians of the branch curves in the quotient, an isomorphism is maintained if these powers of the meridians in the quotient are set equal to the identity.

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